Introduction

In figure 1, several disks of the same size form a tight configuration in their "***circum-rectangle***". By "***tight***", we mean the disks are not able to move.

Our motivation is "How to make the biggest or smallest circum-rectangle for a given number of disks?" We partially answer this question and find several new methods to construct tight configurations of any given number of disks.

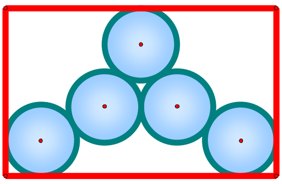


Fig. 1 Tight configuration.

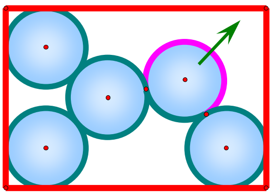


Fig. 2 Free configuration.

Section 1 *Find the circum-rectangles with extreme areas of a few disks.*

Dynamic geometry mathematical software *Geometer's Sketchpad* is used to explore all the possible tight configurations of 2~6 disks. We find the circum-rectangles with extreme areas, as shown in Table 1.

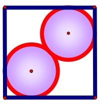
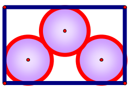
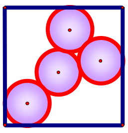
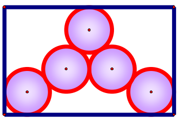
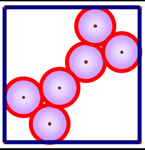
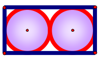
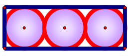
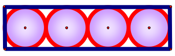
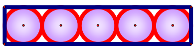
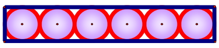
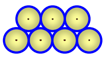
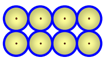
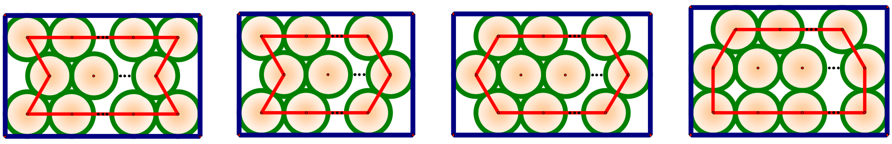
Number of disks	2	3	4	5	6
The maximal area of circum-rectangle	$\left(2+\sqrt{2}\right)^2$ ≈ 11.66	16.65	$\left(2+\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\right)^2$ ≈ 28.58	35.51	$\left(2+2\sqrt{2}+\sqrt{6}\right)^2$ ≈ 52.97
Corresponding circum-rectangle					
The minimal area of circum-rectangle	8	12	16	20	24
Corresponding circum-rectangle					

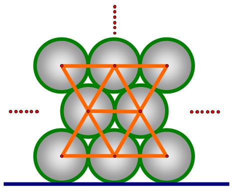
Table 1. The circum-rectangles with extreme areas of 2~6 disks which have radius 1.

Section 2 *Compare the areas of circum-rectangles in linear and equilateral triangular configuration.*

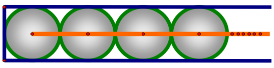
A configuration is said to be equilateral triangular if the configuration can be divided into parallel layers and two adjacent layers are connected in the ways of (*A*)  or (*B*)  (with at least an (*A*)). We take three layers as examples:



Equilateral triangular configuration



Linear configuration



Conclusions

1. The linear configuration results in a smaller area than the equilateral triangular configuration does when there are 2~10, 12, or 13 disks.
2. The equilateral triangular configuration results in a smaller area than the linear configuration does when there are 11, 14 disks or more.

Section 3 *Construct tight configurations with a large number of disks.*

Basic configurations

We discover that the circum-rectangles with maximal areas of 4, 5, 6 disks can be decomposed into *basic configurations*. We also find that the number of basic configurations is infinite. A basic configuration can be placed in a simpler configuration so that we can generate tight configurations with more disks.

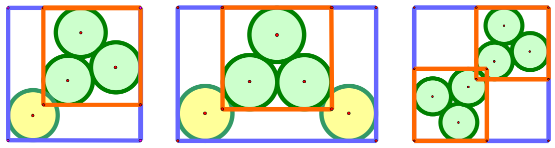


Fig. 3 Basic configurations in circum-rectangles.

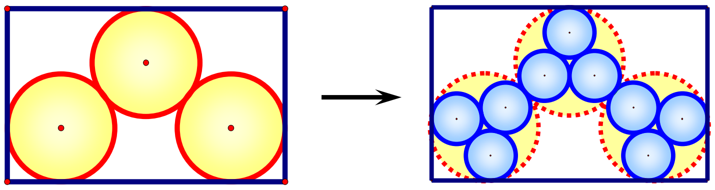


Fig. 4 Placing basic configurations.

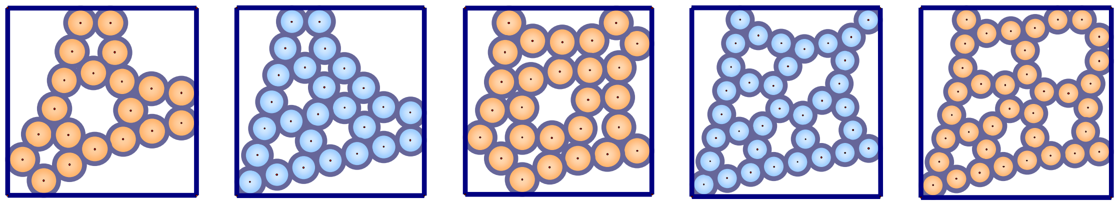


Fig. 5 Basic configurations.

Wave and ring configurations

However, only *wave* and *ring* configurations are able to generate tight configurations. We find that we can place infinite basic configurations in wave configurations and at most eight basic configurations in ring configurations.

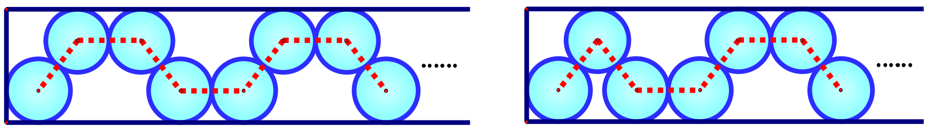


Fig. 6 Wave configurations.

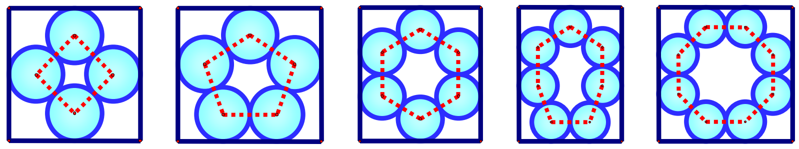
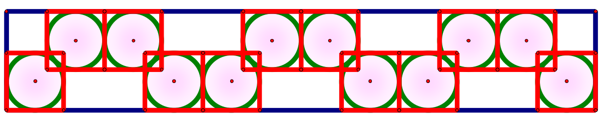
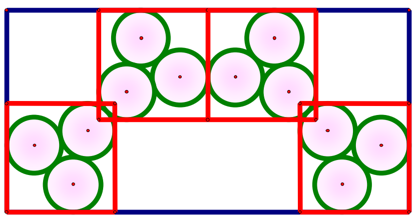
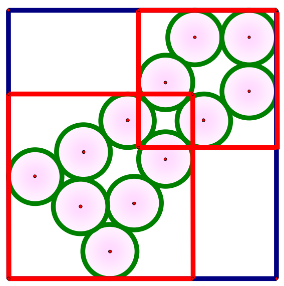
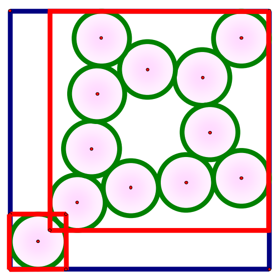


Fig. 7 Ring configurations.

Basic configuration decomposition Method

If we denote a basic configuration by the number of the disks it contains, then a complicated configuration can be represented by a series of natural numbers. We call it the *"Basic configuration decomposition method"*. In this way, we can construct tight configurations of ***any*** number of disks. We take 12 disks as an example:

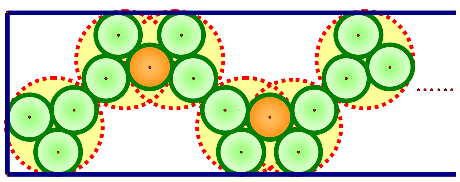
1+1+1+1+1+1+1+1+1+1+1+1	3+3+3+3	5+7	1+11
			

An example of *basic configuration decomposition method*

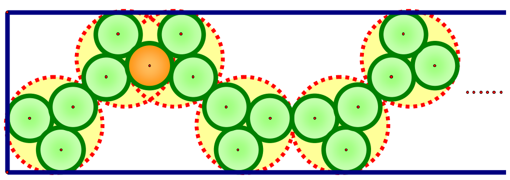
Given $n = 3(2k)-s$ or $3(2k+1)-s$ ($k \in N$, $k \geq 3$, $s = 0, 1, 2$). Here are the corresponding tight configurations:

1. If $n = 3(2k)-s$:

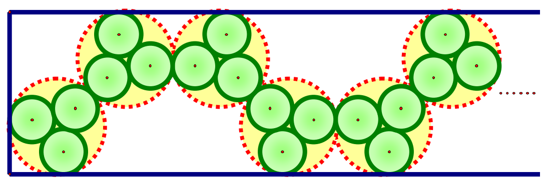
$s = 0$



$s = 1$

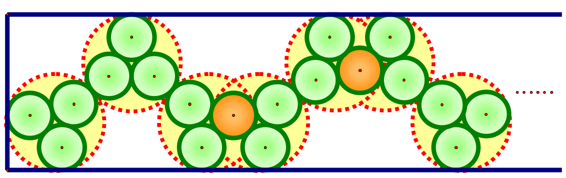


$s = 2$

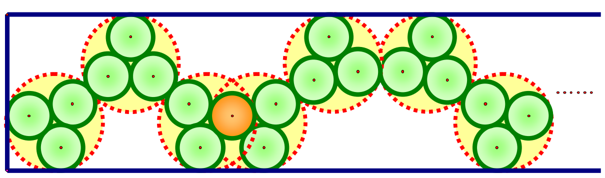


2. If $n = 3(2k+1)-s$:

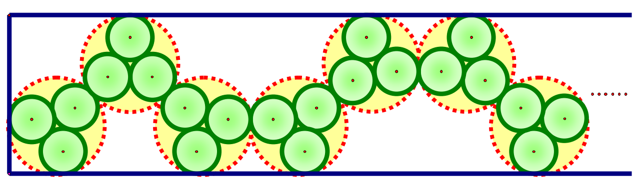
$s = 0$



$s = 1$



$s = 2$



Rearrangement

The lengths of the two adjacent sides of the circum-rectangles of configurations constructed by wave configurations can be quite different. However, we can make the length/width ratio by rearranging appropriate subconfigurations.

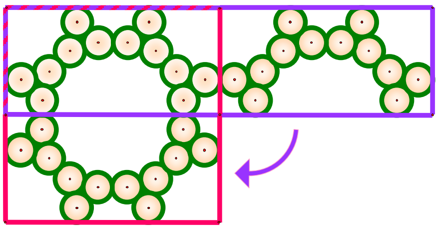
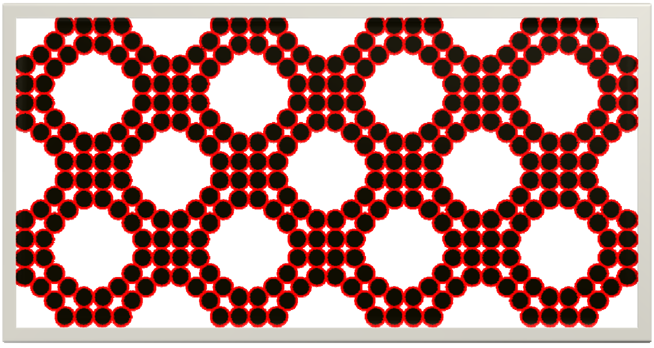
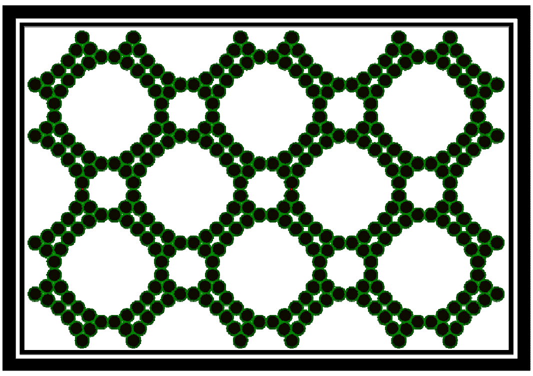


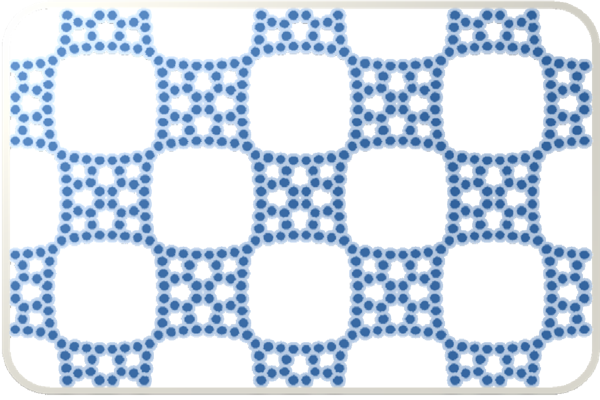
Fig. 8 Rearrangement

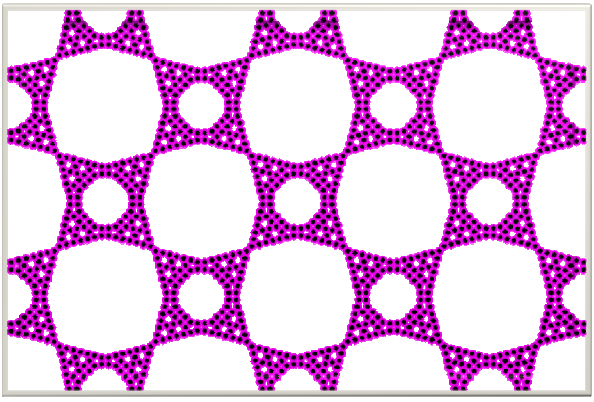
Extension

We can also extend a circum-rectangle by reflections with respect to the edges and get larger tight configurations with beautiful patterns.









Conclusions

1. We find the circum-rectangles that achieve the maximal and minimal areas when there are 2~6 disks.

2. The linear configuration results in a smaller area than the equilateral triangular configuration does when there are 2~10, 12, or 13 disks.

3. By using the *basic configuration decomposition method*, we can construct at least two tight configurations of ***any*** number of disks.

4. For the application, our work may be applied to pattern design, or optimal arrangements of the pipes in a filter or an optical fiber.

References

[1] George G. Szpiro, Kepler's Conjecture, John Wiley & Sons. Inc, 2003.
[2] Ivan Moscovich, Leonardo's Mirror & Other Puzzles, Sterling Publishing, 2005.
[3] Packing problems, Wolfram math world: http://mathworld.wolfram.com/topics/PackingProblems.html
[4] Erich's Packing Center: http://www.stetson.edu/~efriedma/packing.html